① Some Complex Function Theory

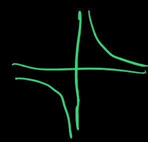
* Holomorphicity

$f: \mathbb{C} \longrightarrow \mathbb{C}$

z_0

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists}$$

* Meromorphicity



$f: \mathbb{C} \to \mathbb{C}$ holomorphic

&

countably many poles $\{p_i\}$

where $f(p_i) = \infty$.

* Complex logarithm

$f(z) = \log z$ | $re^{i\theta}$

$\log r + i\theta + 2\pi i$

* "Logarithmic" derivative

$$\boxed{\frac{f'}{f}} \qquad (\log f)' = \frac{1}{f} \cdot f'$$

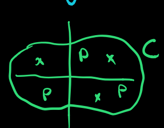
$f(z_0) = 0 \qquad f(z) = (z-z_0)^n \underbrace{g(z)}_{\text{non-vanishing around } z_0}$

$f(z_1) = +\infty$

$$\frac{f'}{f} = \frac{n}{z-z_0} + \cdots$$

$$\frac{f'}{f} = -\frac{n}{z-z_0} + \cdots$$

* Argument Principle



$$\int_C f(z) = 2\pi i \left(\sum \text{res}(p) \; \cancel{\ldots} \right)$$

$$\int_C f'/f = 2\pi i \left(\underline{\# \text{ zeros of } f - \# \text{ poles of } f} \right)$$

② Modular Forms: Definition & First Examples

Notation

Let $\underline{\gamma} \in SL_2(\mathbb{Z}) := \left\{ \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{} : ad - bc = 1,\ a,b,c,d \in \mathbb{Z} \right\}$

Let $\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$

$\gamma\tau = \dfrac{a\tau + b}{c\tau + d}$

* Definition

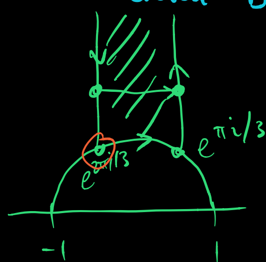
$f : \mathbb{H} \longrightarrow \mathbb{C}$ $\underline{\text{modular of weight } k}$: $f(\gamma\tau) = f(\tau) \cdot (\underline{c}\tau + \underline{d})^k$

$k = 0$: $\underline{f(\gamma\tau)} = \underline{f(\tau)}$ $\qquad f(\gamma\tau) = f(\tau)\,(-1)^k \Rightarrow f(\tau) = \underbrace{f(\tau)(-1)^k}_{}$

* Litmus Test for Modularity

$\langle S, T \rangle = SL_2(\mathbb{Z}) \qquad S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

* Fundamental Domain



$\tilde{\mathcal{F}}$ = fundamental domain

$\mathcal{D}$ = boundary on right-half plane $\cup \{e^{2\pi i/3}\}$

* Example: Eisenstein Series

Let $k \geq 4$, k even.

$$G_k(\tau) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m + n\tau)^k}$$

$\boxed{m + n\tau \longmapsto (m+n) + n\tau}$

E_4, E_6 generate all modular forms of positive weight.

③ The Valence Formula (i.e., the base case)

Thm: Let $\mathcal{F}$ and $\mathcal{D}$ be as above, and let f be a nonzero modular form of weight k. Then,

$$\frac{k}{12} = \underbrace{V_\infty(f)}_{} + \frac{1}{2}\underbrace{V_i(f)}_{} + \frac{1}{3} V_{e^{2\pi i/3}}(f) + \sum_{\tau \in \mathcal{F} \setminus \mathcal{D}} V_\tau(f).$$

$M_k(SL_2(\mathbb{Z})) := \{\text{modular forms of weight } k\}$

1st goal: prove that $M_k(SL_2(\mathbb{Z}))$ is finite-dimensional.

* $k=4$: $\frac{1}{3} = \_\ \_\ \_\ \_\ \_$

  $\Rightarrow$ Can only vanish at $e^{2\pi i/3}$

  $\dim_{\mathbb{C}} M_4(SL_2(\mathbb{Z})) = 1$

* $k=6$: $\frac{1}{2} = \ .\ .\ .\ .$

  $\Rightarrow$ Can only at i

  $\dim_{\mathbb{C}} M_6(SL_2(\mathbb{Z})) = 1$

* $k=10$: $\frac{5}{6}$

③ The Valence Formula (i.e., the base case)

<u>Thm:</u> Let $\mathcal{F}$ and $\mathcal{D}$ be as above, and let f be a nonzero modular form of weight k. Then,

$$\frac{k}{12} = v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_{e^{2\pi i/3}}(f) + \sum_{\tau \in \mathcal{F} \setminus \mathcal{D}} v_\tau(f).$$

* First Dimensionality Results

$$\dim_{\mathbb{C}}(M_k(SL_2(\mathbb{Z}))) = \begin{cases} 0 & k \text{ odd}, k=2 \\ 1 & k = 0, 4, 6, 8, 10 \end{cases}$$

3 The Valence Formula (i.e., the base case)

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* The Dimension Formula

$$\dim_{\mathbb{C}}(M_k(SL_2(\mathbb{Z}))) = \begin{cases} \lfloor k/12 \rfloor & k \equiv 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & k \not\equiv 2 \pmod{12} \end{cases}$$

$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$

$$M_{k-12}(SL_2(\mathbb{Z})) \longrightarrow S_k(SL_2(\mathbb{Z}))$$

$$f \longmapsto f\underbrace{\Delta}_{}$$

cusp forms

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* Basis of $M_k(SL_2(\mathbb{Z}))$

$$M_k(SL_2(\mathbb{Z})) = \langle \{ E_4^{\,a} E_6^{\,b} : 4a + 6b = k;\ a, b \in \mathbb{Z}_{\geq 0} \} \rangle$$

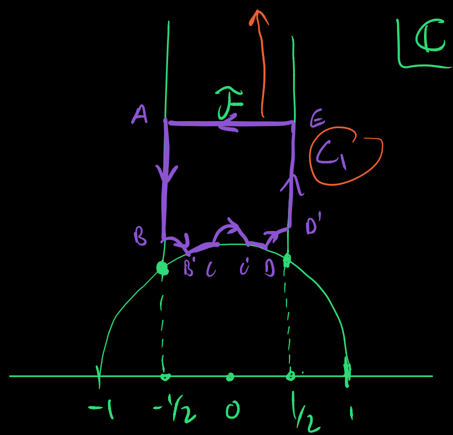
$\mathbb{C}[E_4, E_6]$ = "all modular forms of positive or zero weight"

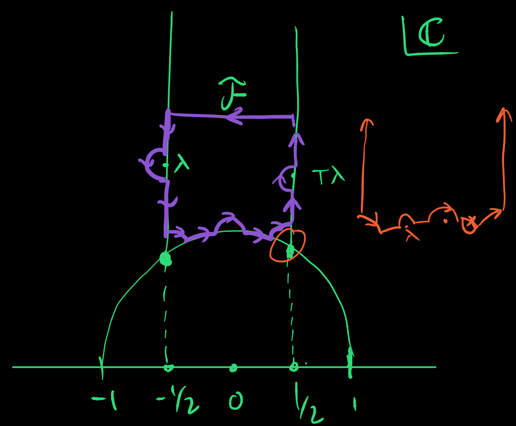
$$\updownarrow$$

$\mathbb{C}[x, y]$

4a Proof of Valence: Complex Analysis on $\mathbb{C}$

* Choice of Contour





Case 1: Zeroes $\in \{e^{2\pi i/3}, i, e^{\pi i/3}\}$

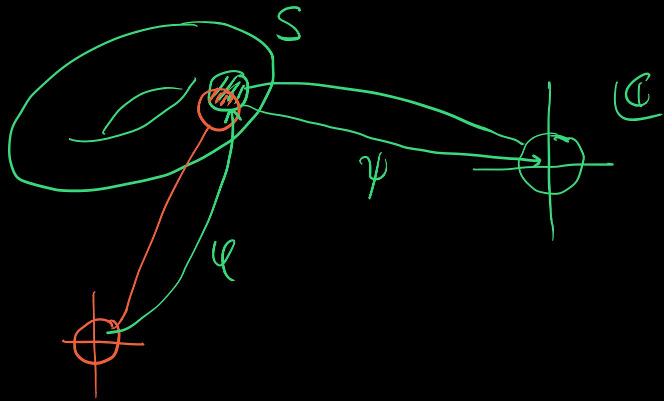
f modular form:

$$\int_{C_1} \frac{f'}{f} = \sum_{p \in \mathcal{F} - D} v_p(f)$$

Case 2: A ~~Zero~~ Zero λ on the Vertical Lines

(4b) (Brief) Proof of Valence: Riemann Surfaces

* Definition of Riemann Surface



1) Collection of charts which map biholomorphically to an open set in $\mathbb{C}$

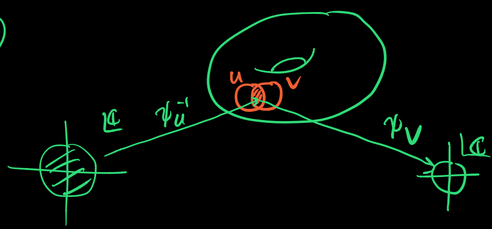
2) $\psi \circ \varphi$ holomorphic

4b (Brief) Proof of Valence: Riemann Surfaces

* Definition of Riemann Surface

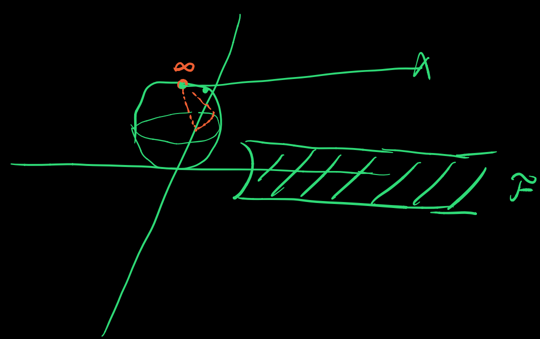
1

2



$\psi_V \circ \psi_U^{-1}$ is holomorphic and 1-to-1.

* Stereographic Projection

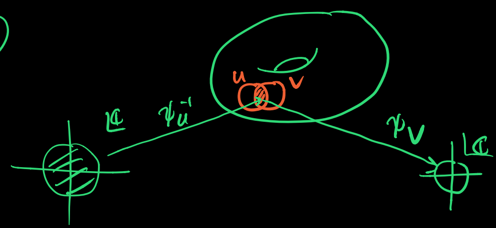


(4b) (Brief) Proof of Valence: Riemann Surfaces

* Definition of Riemann Surface

(1)

(2)



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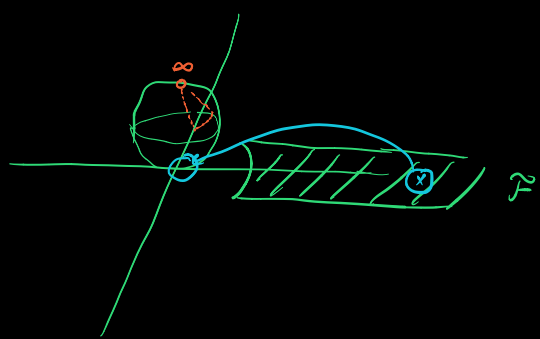
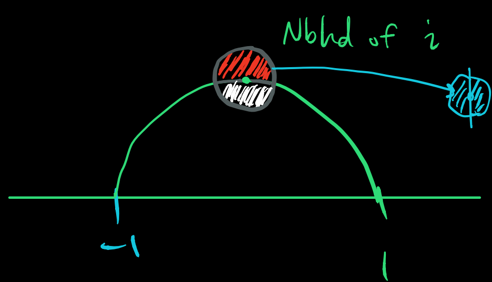


$\xrightarrow{\text{compactification at } \infty}$ $\mathcal{F} \cup \{\infty\}$ becomes a Riemann surface, called $X(1)$

* Charts



4b (Brief) Proof of Valence: Riemann Surfaces

* Symmetric k-Forms on X (Riemann surface): $\Omega^{(k)}(X) = \{ f(z)(dz)^k : f \text{ is meromorphic on } X \}$

$$\int \boxed{f(z)\,dz} \qquad f(z)(dz)^k$$

$$\operatorname{div}(\omega) := \sum_{x \in X} \operatorname{ord}_x(f) \cdot x$$

$$\deg(\operatorname{div}(\omega)) = \sum_{x \in X} \operatorname{ord}_x(f)$$

$\operatorname{ord}_x f = v_0(g)$

* Riemann–Roch Theorem

$$\deg(\operatorname{div}(\omega)) = (2g-2) \cdot k$$

* Fixed Forms of $\Omega^{(k)}(X)$ under $SL_2(\mathbb{Z})$

For $f \in \mathcal{M}_k(SL_2(\mathbb{Z}))$, $f(z)(dz)^k$ is fixed under transformation by $SL_2(\mathbb{Z})$, and so is a k-form on X.

Then, there exists a form ω_f on $\Omega^{(k)}(X(1))$ such that $\omega_f \circ \phi = f(z)(dz)^k$, where $\phi : \mathbb{H} \longrightarrow X(1)$, $\tau \longmapsto SL_2(\mathbb{Z})\tau$.

$$\phi(\tau_x) = x$$

$$\operatorname{ord}_{\tau_x}(f)$$

4b (Brief) Proof of Valence: Riemann Surfaces

* Symmetric k-Forms on X: $\Omega^{(k)}(X)$

$$\Omega^{(k)}(X) := \{ f(z)(dz)^k : f \text{ meromorphic on } X \}$$

$$\operatorname{div}(\omega) := \sum_{x \in X} \operatorname{ord}_x(f) \cdot x \quad , \quad \omega = f(z)(dz)^k$$

$$\deg(\operatorname{div}(\omega)) := \sum_{x \in X} \operatorname{ord}_x(f)$$

* Riemann–Roch Theorem

$$\underline{\deg(\operatorname{div} \omega)} = k(2g-2)$$

* Form on $\mathbb{H}$ $\longrightarrow$ Form on $X(1)$

For $f \in \mathcal{M}_k(SL_2(\mathbb{Z}))$, $f(z)(dz)^k$ is fixed under transformation by $SL_2(\mathbb{Z})$, and so is a k-form on X.

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$$\tau \longmapsto SL_2(\mathbb{Z})\tau .$$

* The Valence Formula

$$\phi(\tau_x) = x$$

$$\operatorname{ord}_{\tau_x}(f) = \begin{cases} \operatorname{ord}_x(f) & \phi(\tau_x) \neq i, e^{2\pi i/3}, \infty \\ \frac{1}{2}\operatorname{ord}_i(f) - k/2 & \phi(\tau_x) = i \\ \frac{1}{3}\operatorname{ord}_{e^{2\pi i/3}}(f) - k/3 & \text{"} \quad = e^{2\pi i/3} \\ \operatorname{ord}_\infty(f) - k & \text{"} \quad = \infty \end{cases}$$

$$\deg(\mathrm{div}(\omega)) = -2k$$

$$-2k = \left(\underline{\mathrm{ord}_\infty(f)} - k\right) + \left(\underline{\tfrac{1}{3}\mathrm{ord}_{e^{2\pi i/3}}}(f) - k/3\right)$$

$$+ \left(\underline{\tfrac{1}{2}\mathrm{ord}_i}(f) - k/2\right)$$

$$\underline{+ \sum_{\substack{x \in X \\ \phi(x) \neq i, \infty, e^{2\pi i/3}}} \mathrm{ord}_x(f)}$$

$$\frac{k}{6} = \ldots$$

$$\frac{k}{6} = \left(\mathrm{ord}_\infty(f) + \ldots\right.$$